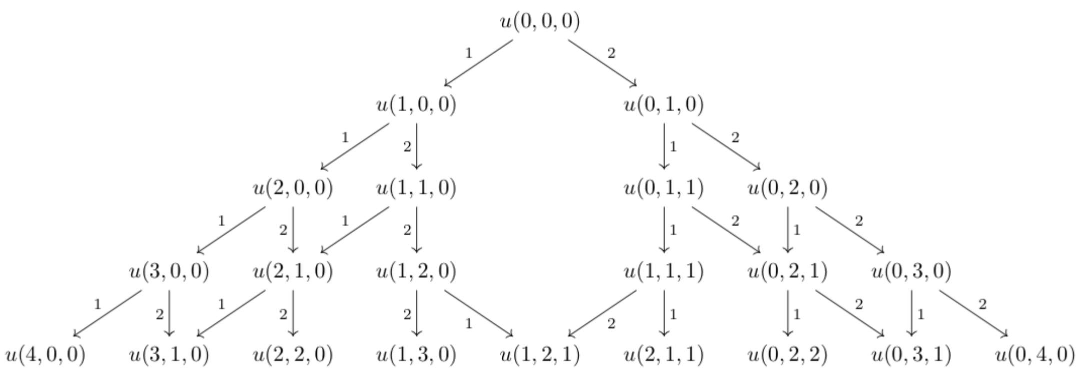
THE $\mathcal{B}(\infty)$ CRYSTAL



IN THE DEFINITION OF A CRYSTAL I ASSUMED

✓ $\varepsilon_i(v) = \max\{n \mid e_i^n(v) \neq 0\}$ (*)

... $\cancel{\varphi_i(v) = \max\{\ell \mid f_i^\ell(v) \neq 0\}}$ (**)

$$\underbrace{\bullet \longleftarrow \underset{f_i v}{\bullet} \longleftarrow}_{\ell = \varphi_i(v)} \underbrace{\underset{v}{\bullet} \longleftarrow \underset{e_i v}{\bullet} \longleftarrow \underset{e_i^2 v}{\bullet} \longleftarrow \underset{e_i^n v}{\bullet}}_{h = \varepsilon_i(v)}$$

AND $\varphi_i(v) - \varepsilon_i(v) = \langle wt(v), \alpha_i^\vee \rangle$. (***)

WE MUST (FOLLOWING KASHIWARA) RELAX (*) AND (**).

WE STILL NEED $(***)$.

BUT WE CAN ASSUME $(*)$ BUT NOT $(**)$. IF $(*)$ AND $(**)$ ARE TRUE

CRYSTAL IS CALLED SEMINORMAL.

IF ONLY $(*)$ IS TRUE SAY $\mathcal{C}$ IS UPPER SEMINORMAL. EVEN ALLOW φ_i TO TAKE NEGATIVE VALUES.

ALWAYS
$$\varepsilon_i(e_i(x)) = \varepsilon_i(x) - 1$$
$$\varphi_i(e_i(x)) = \varphi_i(x) + 1$$
$$wt(e_i(x)) = wt(x) + \alpha_i$$
$$wt(f_i(x)) = wt(x) - \alpha_i.$$

EXAMPLE: IF $\mathcal{C}$ IS A CRYSTAL

REDEFINE
$$\tilde{\varphi}_i(x) = \varphi_i(x) + 100.$$
$$\tilde{\varepsilon}_i(x) = \varepsilon_i(x) + 100$$

EXAMPLE: IF I HAVE A CRYSTAL $\mathcal{C}$

$\mu \in \Lambda$ (WEIGHT LATTICE $= \mathbb{Z}^n$ FOR $GL(n)$ CRYSTALS)

$\mathcal{C}' = \mathcal{C}$ WITH MODIFICATIONS

$$wt'(v) = wt(v) + \mu$$

$$\varphi_i'(v) = \varphi_i(v) + \langle \mu, wt(v) \rangle$$

$$\varepsilon_i'(v) = \varepsilon_i(v)$$

STILL A CRYSTAL. (NOT NEC. SEMINORMAL)

STILL UPPER SN IF $\mathcal{C}$ IS.

WE CAN SHIFT THE WEIGHT.

$$\mathcal{C}' = \mathcal{C} \otimes T_\mu$$

T_μ IS A CRYSTAL WITH ONE ELT

OF WEIGHT μ. TENSORING WITH T_μ

JUST SHIFTS THE WEIGHT AS ABOVE.

WHAT IS $\mathcal{B}(\infty)$?

IT IS UPPER SN

$$\varepsilon_i(x) = \max\{n \mid e_i^n x \neq 0\} \geq 0$$

IT HAS A UNIQUE "HIGHEST WEIGHT EL."

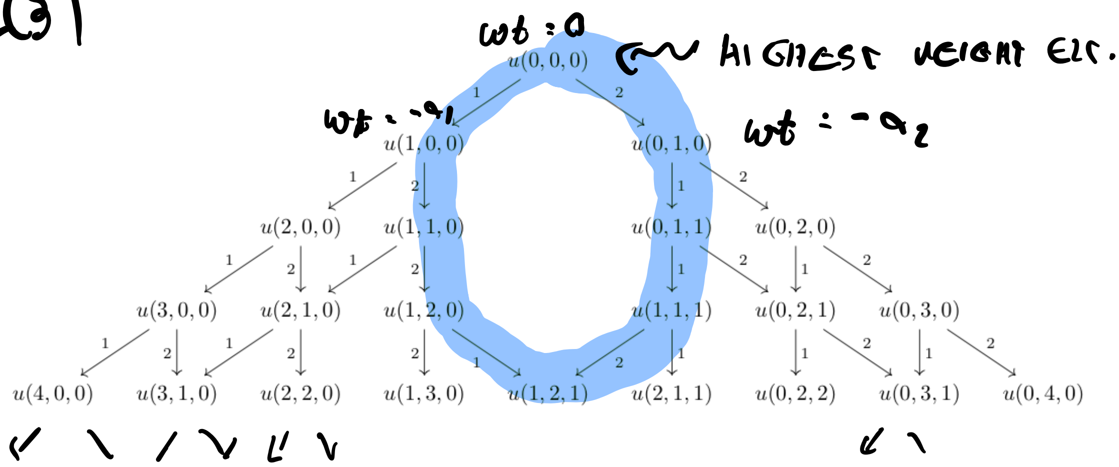
u_0 AND $wt(u_0) = 0$

$\mathcal{B}(\infty)$ CONTAINS ALL THE CRYSTALS WE'VE ALREADY STUDIED MORE PRECISELY

$\mathcal{B}_\lambda$ λ A DOMINANT WEIGHT.

$\mathcal{B}(\infty) \otimes T_\lambda$ CONTAINS A COPY OF $\mathcal{B}_\lambda$.

GL(3)



$$\begin{tikzcd}
 & \begin{array}{ll} 1 & 1 \\ 2 & \end{array} \arrow[dl, "1"'] \arrow[dr, "2"] & \\
\begin{array}{ll} 1 & 2 \\ 2 & \end{array} \arrow[d, "2"'] & & \begin{array}{ll} 1 & 1 \\ 3 & \end{array} \arrow[d, "1"] \\
\begin{array}{ll} 1 & 3 \\ 2 & \end{array} \arrow[d, "2"'] & & \begin{array}{ll} 1 & 2 \\ 3 & \end{array} \arrow[d, "1"] \\
\begin{array}{ll} 1 & 3 \\ 3 & \end{array} \arrow[dr, "1"'] & & \begin{array}{ll} 2 & 2 \\ 3 & \end{array} \arrow[dl, "2"] \\
 & \begin{array}{ll} 2 & 3 \\ 3 & \end{array} . &
\end{tikzcd}$$

ADJOINT SQUARE CRYSTAL

$\mathcal{B}_{(2,1)}$ FOR GL(3)

$\mathrm{wt}\begin{pmatrix} 1 & 1 \\ 2 & \end{pmatrix} = (2,1,0)$

$\mathcal{B}(\infty) \otimes T_{(2,1,0)}$

IDEAS FROM REP'N THEORY THAT "EXPLAIN" $B(\infty)$.

LIE ALGEBRA OF $GL(n, \mathbb{C})$ "G" IS

$\mathfrak{g} = MAT(n, \mathbb{C})$ WITH SOME ALGEBRAIC STRUCTURE

$X, Y \in \mathfrak{g}$ THERE IS DEFINED

$$[X, Y] = XY - YX$$

IF I HAVE A REP'N

$$\pi : G \to GL(V)$$

AND $X \in \mathfrak{g}$ DEFINE

$$d\pi(X)v := \frac{d}{dt} \pi(e^{tX})v \Big|_{t=0}$$

$$d\pi[X, Y] = [d\pi(X), d\pi(Y)]$$

CAN BE PROVED SO OBTAIN

A REP'N OF THE LIE ALGEBRA.

$e_i = d\pi(E_{i,i+1})$ $\qquad f_i = d\pi(E_{i+1,i})$

1 IN $i, i+1$ POSITION
0 ELSEWHERE

MIMIC CRYSTAL OPS

$$wt(e_i(x)) = wt(x) + \alpha_i$$
$$wt(f_i(x)) = wt(x) - \alpha_i.$$

REP OF G $\rightsquigarrow$ REP OF $\mathfrak{g}$

$\leftsquigarrow$
NOT ALWAYS.

GIVEN A REP'N OF G WEIGHT MULTIPLICITIES ARE W-INVARIANT

$W = S_n$ = WEYL GROUP.

ACTS ON Λ.

BECAUSE W CAN BE EMBEDDED IN G AND PERMUTES WEIGHT SPACES

$$V_\mu = \{ v \in V \mid \pi(t) v = t^\mu \cdot v \}.$$

t DIAGONAL

$$t = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \qquad t^\mu = t_1^{\mu_1} \cdots t_n^{\mu_n}$$

$$\mu \in \Lambda = \mathbb{Z}^n$$

$$V_\mu = \{ v \in V \mid d\pi(t) = \langle t, \mu \rangle v \}$$

FOLLOWS FROM ABOVE BY CALCULUS.

$$t = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \qquad \langle t, \mu \rangle = \sum \mu_i t_i$$

CANNOT EMBED W IN LIE ALGEBRA

SO ARGUMENT THAT

DIM $V_{\omega(\mu)}$ = DIM V_μ

BREAKS DOWN.

$w \in W$ $\qquad$ $w(V_\mu) = V_{w(\mu)}$.

EXAMPLE: $\mathfrak{g} = \mathfrak{gl}_2$

λ A DOMINANT WEIGHT $\lambda_1 \geq \lambda_2$

FOR $GL(2)$ THERE IS A REP'N $\vee^n \mathbb{C}^2$

WITH THESE WEIGHTS

(FIX $n > 0$)

(λ_2, λ_1) $\quad \bullet \quad \cdots \quad \bullet \quad$ $(\lambda_1 - 1, \lambda_2 + 1)$ $\bullet$ $\quad$ (λ_1, λ_2) $\bullet$

$\lambda - \alpha$

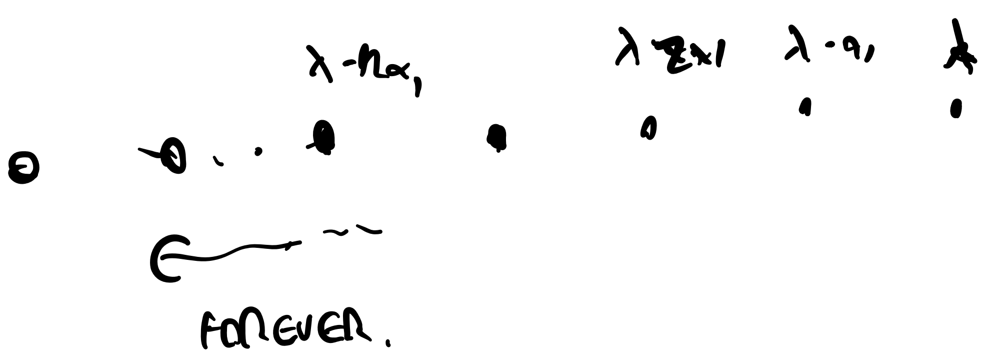
$\lambda - n\alpha$

$\alpha = (1, -1)$

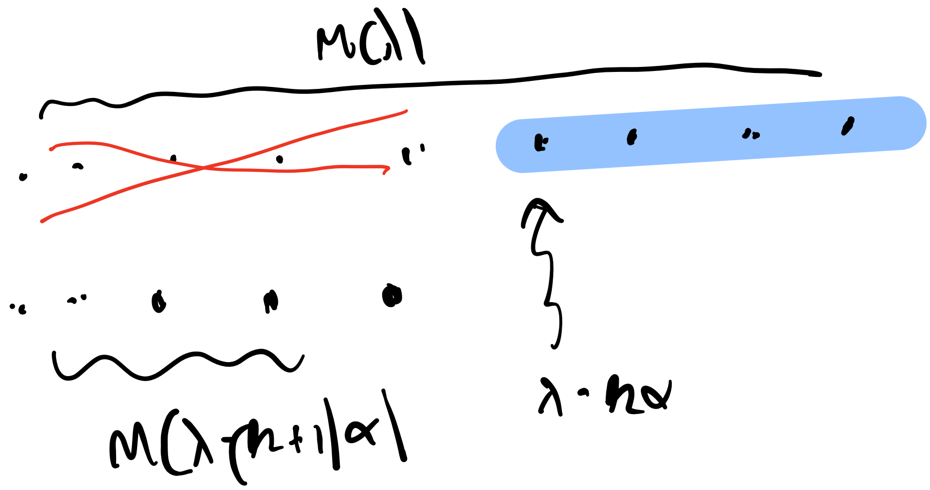
$n = \lambda_1 - \lambda_2$.

THERE EXIST ALSO A REP'N OF $GL_2(\mathbb{C})$ CALLED A "VERMA MODULE" $M(\lambda)$ WITH WEIGHTS

$\lambda, \lambda - \alpha_1, \lambda - 2\alpha_1, \dots$



THIS CONTAINS A COPY OF $M(\lambda - (n+1)\alpha)$ AS A SUBMODULE.



$$M(\lambda)/M(\lambda-(k+1)\alpha) \cong v^k \mathbb{C}^2.$$

THIS FACT EXPLAINS WHY THERE IS A $\mathcal{B}(\infty)$ CRYSTAL; IT IS THE CRYSTAL OF THE VERMA MODULE. V_0

$$V_\mu = V_0 \otimes \mu$$

THERE IS A BIJECTION $V_0 \to V_\lambda$ THAT SHIFTS THE WEIGHT OF EVERY VECTOR.

$\Delta_\lambda(x_1, \dots, x_n)$ CHARACTER OF V_λ

$$\overset{\shortparallel}{\sum_\mu} \dim(V_\mu) \cdot x^\mu \qquad x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$$

$$A\left(\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \middle| V_\lambda\right) = \Delta_\lambda$$

$$\Delta_\lambda(x_1, \dots, x_n) = \frac{\begin{vmatrix} x_1^{\lambda_1+n-1} & x_2^{\lambda_1+n-1} & \cdots \\ x_1^{\lambda_2+n-2} & x_2^{\lambda_2+n-2} & \\ \vdots & & \\ x_1^{\lambda_n} & x_2^{\lambda_n} & \cdots \end{vmatrix}}{\begin{vmatrix} x_1^{n-1} & x_2^{n-1} & \cdots \\ x_1^{n-2} & x_2^{n-2} & \\ \vdots & \vdots & \\ 1 & 1 & \end{vmatrix}}$$

DENOM $\displaystyle\prod_{i<j} (x_i - x_j)$

(VANDERMONDE DETERMINANT IDENTITY.

NUMERATOR $\quad \rho = (n-1, n-2, \dots, 0)$

$$x_1^{\lambda_1+n-1} x_2^{\lambda_2+n-2} \cdots x_n^{\lambda_n}$$

DIAGONAL IN $|\cdots|$ EQUALS

$$x^{\lambda+\rho}$$

$$\text{NUMERATOR} = \sum_{w \in W} \det(w)\, x^{w(\lambda+\rho)}$$

$$\Delta_\lambda(x) = \frac{\sum_{w \in W} \det(w)\, x^{w(\lambda+\rho)}}{\sum_{w \in W} \det(w)\, x^{w(\rho)}}$$

DIVIDE NUMERATOR AND DENOM BY x^ρ

$$\frac{\sum_{w \in W} \det(w)\, x^{w(\lambda+\rho)-\rho}}{\sum_{w \in W} \det(w)\, x^{w(\rho)-\rho}}$$

DENOMINATOR IS

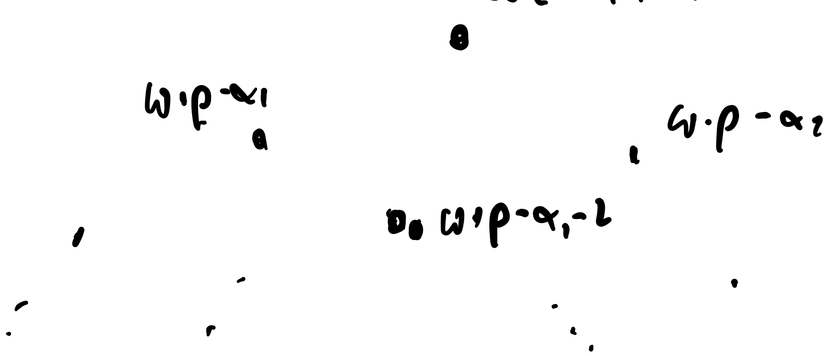
$$\prod_{i<j}\left(\frac{x_i - x_j}{x_i}\right) = \prod_{i<j} 1 - \frac{x_j}{x_i}$$

$$\Delta_\lambda(x) = \sum_{w\in W} \det(w)\, x^{w(\lambda+\rho)-\rho} \underbrace{\prod_{i<j}\left(1 - \frac{z_j}{z_i}\right)^{-1}}$$

CHARACTER OF A $\mathfrak{g}$-MODULE; THE VERMA MODULE

$M(w(\lambda+\rho)-\rho)$.

$$w(\lambda+\rho)-\rho = w\cdot\lambda$$



$$\mathrm{ch}\, V_\lambda = \sum_{w \in W} \det(w)\ \mathrm{ch}\, M(w \cdot \lambda)$$

$$w(\lambda+\rho)-\rho.$$

SIMPLEST CASE $n = 2$

$$0 \to M(\lambda - (n+1)\alpha) \to M(\lambda) \to V_\lambda \to 0$$

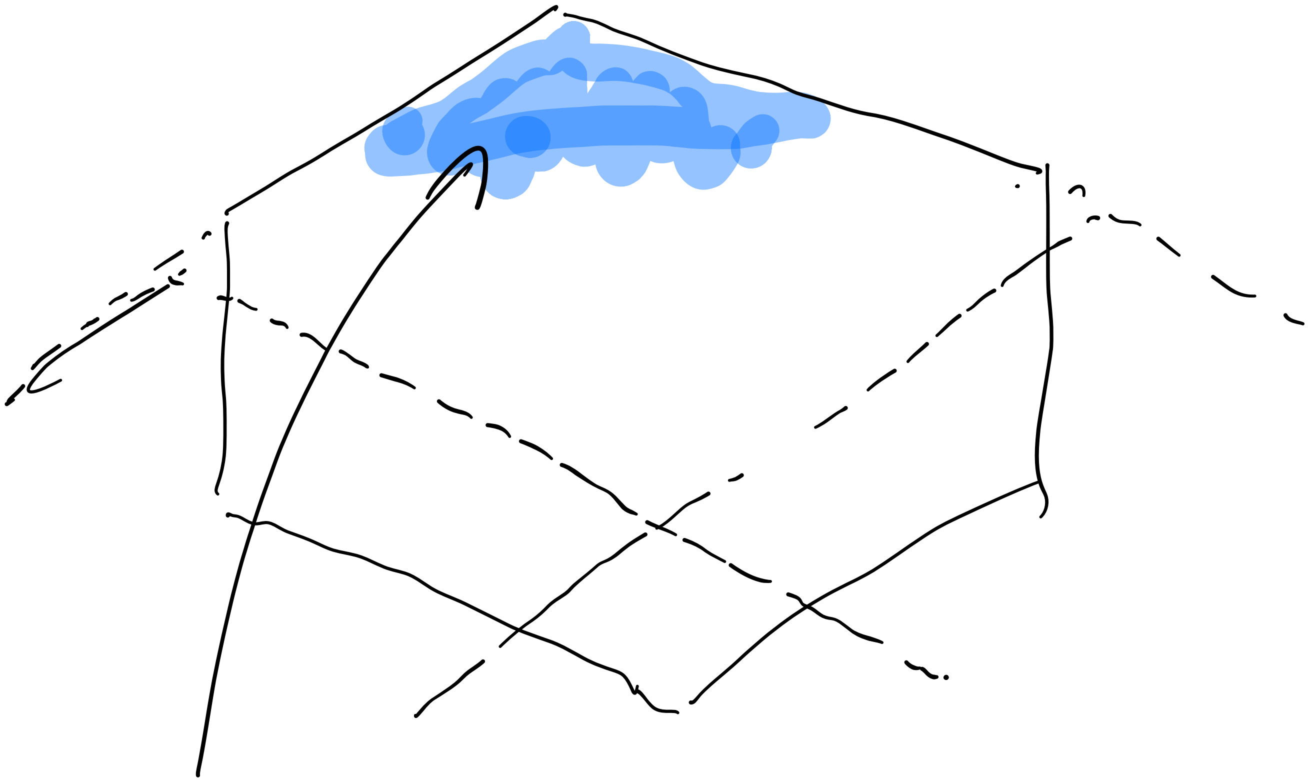
$$n = (\lambda, \alpha)$$

VERMA MODULE $M(\lambda)$ HAS A SIMPLE CHARACTER

$$\prod_{i<j} \left(1 - \frac{z_j}{z_i}\right)^{-1}$$

TO VISUALIZE $\mathcal{B}(\infty)$ TAKE

$\lambda_1 > \lambda_2 > \ldots$ $\qquad \lambda_i - \lambda_{i+1}$ LARGE.



THIS PORTION OF THIS CRYSTAL WILL LOOK LIKE $\mathcal{B}(\infty)$.

THIS APPROACH TO CONSTRUCTING $\mathcal{B}(\infty)$ IS CALLED "MARGINALLY LARGE TABLEAUX" GOOD ALGORITHM (IT IS IN SAGE).

FOR λ A DOM. WT.

CONSTRUCT A CRYSTAL $\mathcal{B}_\lambda$ AND AN IRREP (π_λ, V_λ) WANT TO KNOW

$$\operatorname{ch} \mathcal{B}_\lambda = \sum_{v \in \mathcal{B}_\lambda} z^{wt(v)} = \operatorname{ch} V_\lambda.$$

NEED DEMAZURE CRYSTALS AND $\mathcal{B}(\infty)$.